Mathematical Induction

Part Two

Outline for Today

- ***Variations on Induction***
  - Starting later, taking different step sizes, and more!
- ***"Build Up" versus "Build Down"***
  - An inductive nuance that follows from our general proofwriting principles.
- ***Complete Induction***
  - When one assumption isn't enough!

Recap from Last Time

Let P be some predicate. The ***principle of mathematical induction*** states that if

If it starts true…

$P(0)$ is true

and

…and it stays true…

$$\forall k \in \mathbb{N}.\ (P(k) \to P(k+1))$$

then

$$\forall n \in \mathbb{N}.\ P(n)$$

…then it's always true.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: Let $P(n)$ be the statement "the sum of the first n powers of two is $2^n - 1$." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

New Stuff!

Variations on Induction: ***Starting Later***

Induction Starting at 0

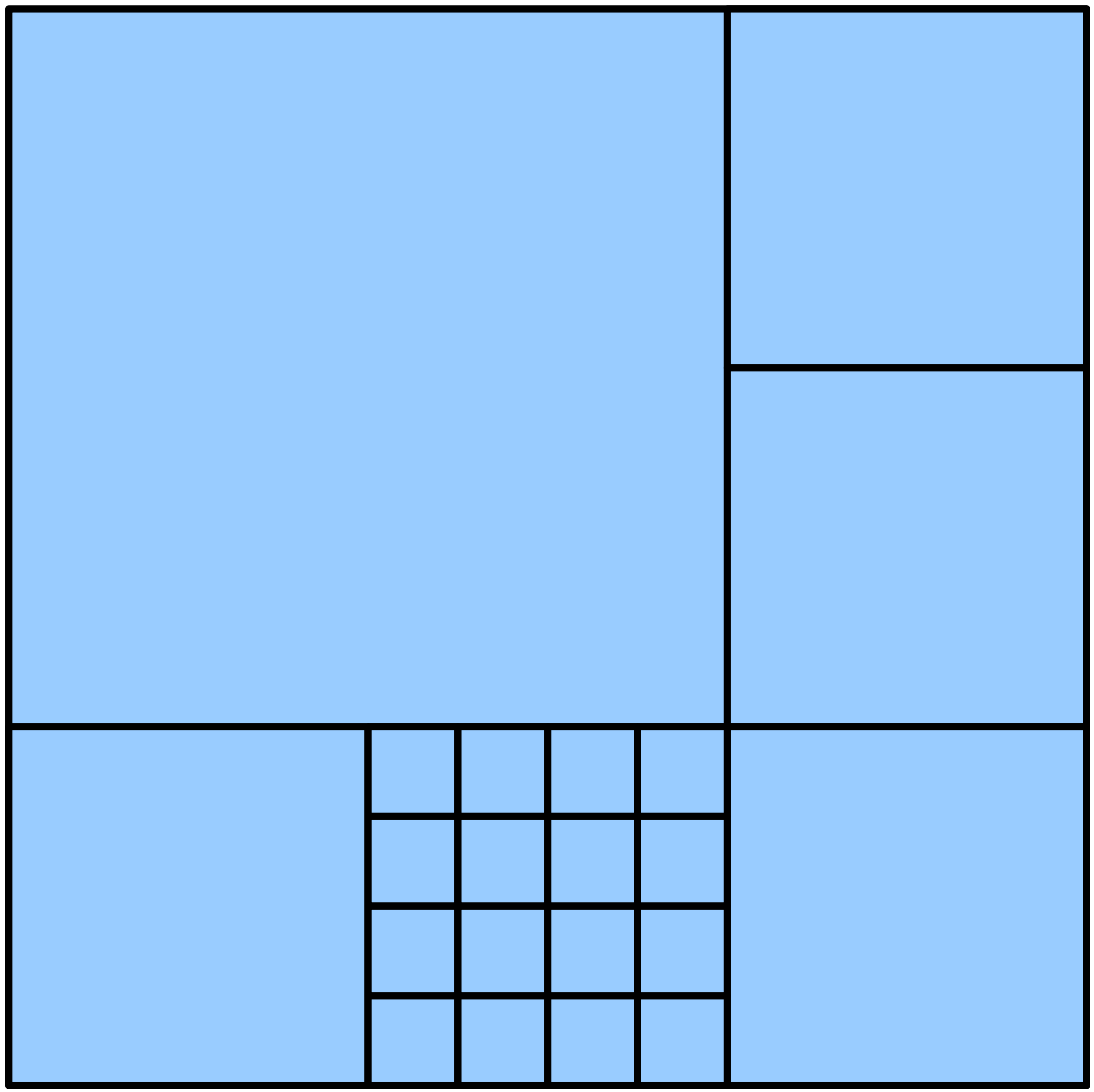
- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  - Show that $P(0)$ is true.
  - Show that for any $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at $\boldsymbol{m}$

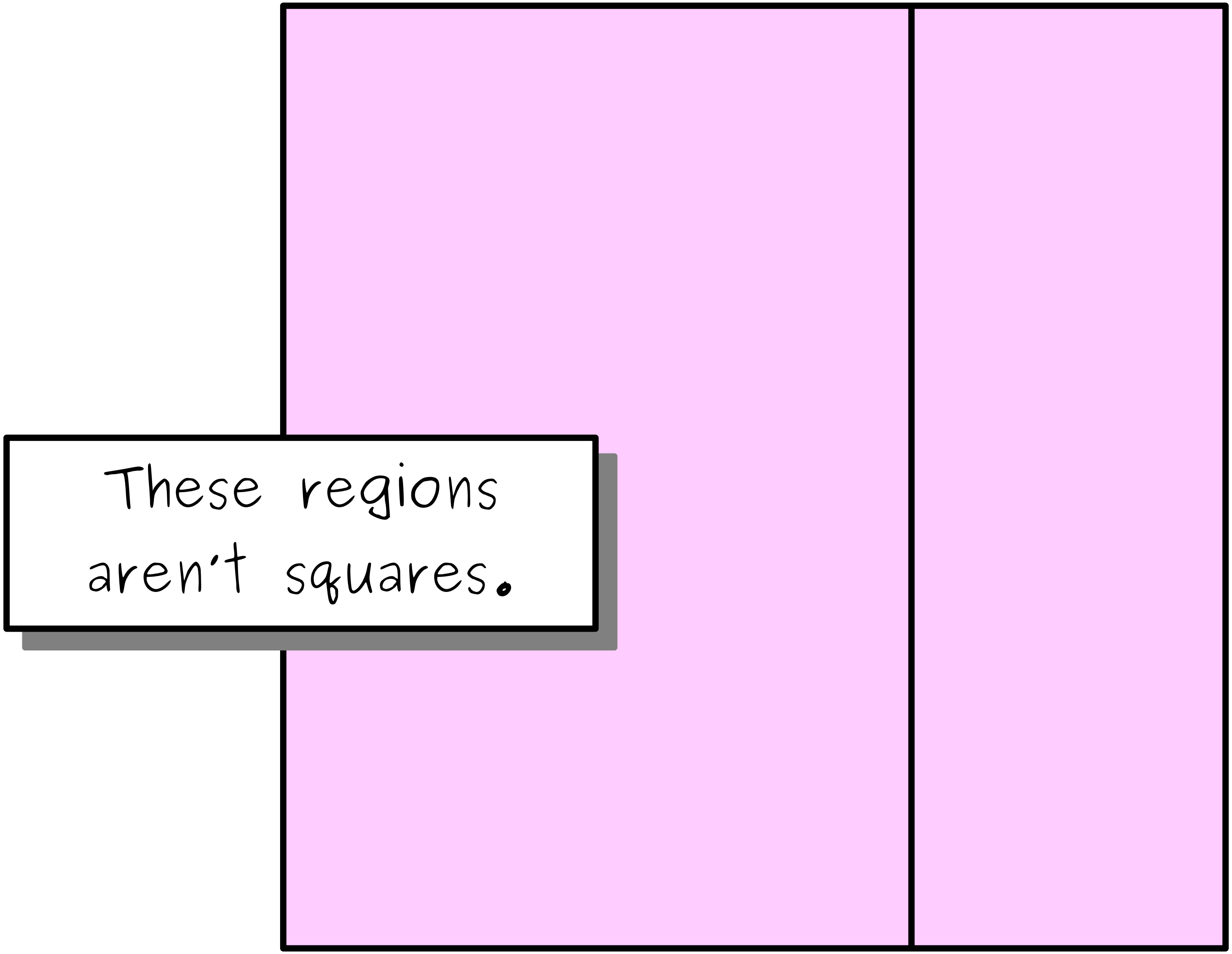
- To prove that $P(n)$ is true for all natural numbers greater than or equal to $\boldsymbol{m}$:
  - Show that $P(\boldsymbol{m})$ is true.
  - Show that for any $k \geq \boldsymbol{m}$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to $\boldsymbol{m}$.

Variations on Induction: ***Bigger Steps***

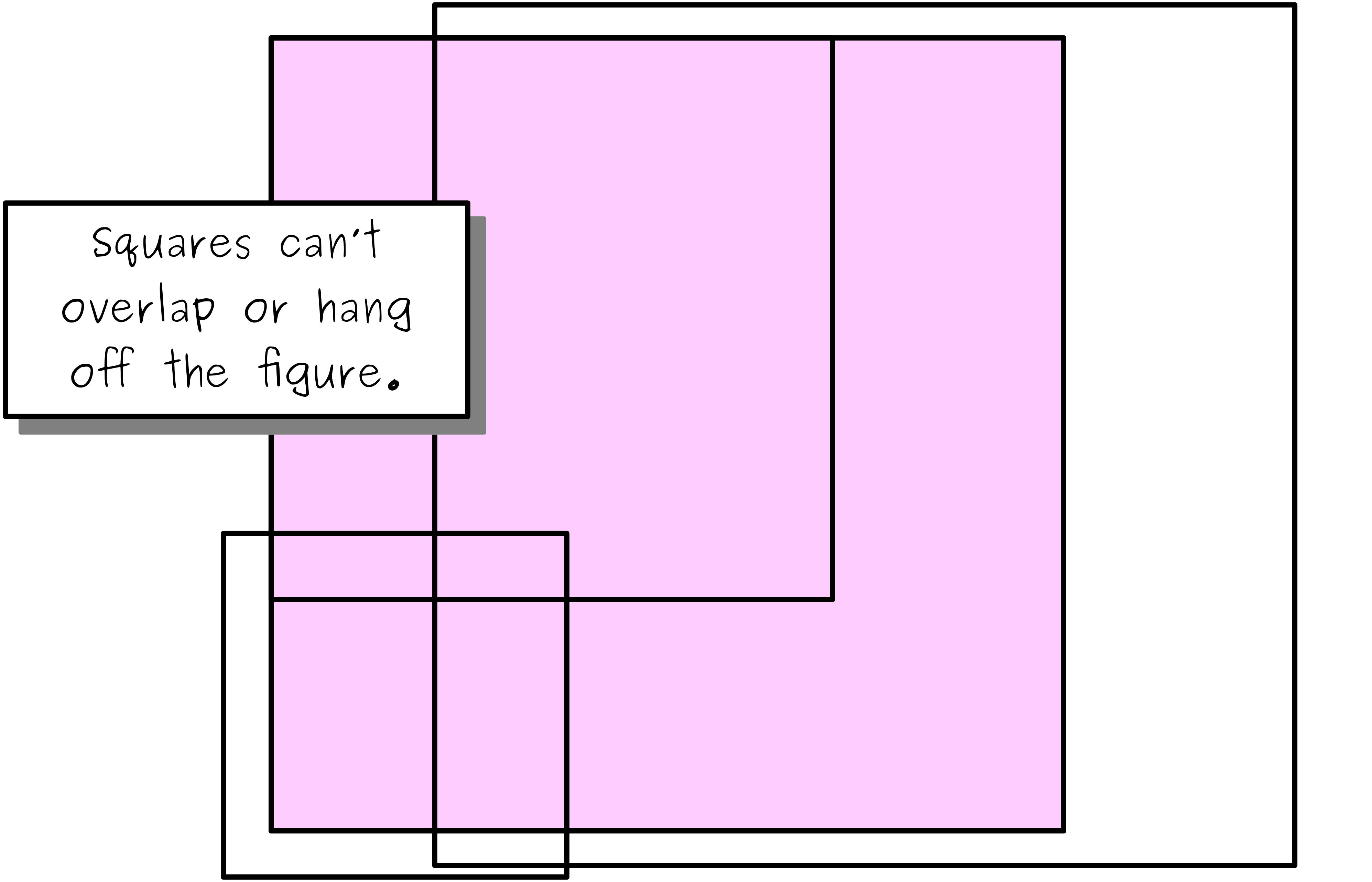
Subdividing a Square



Subdividing a Square



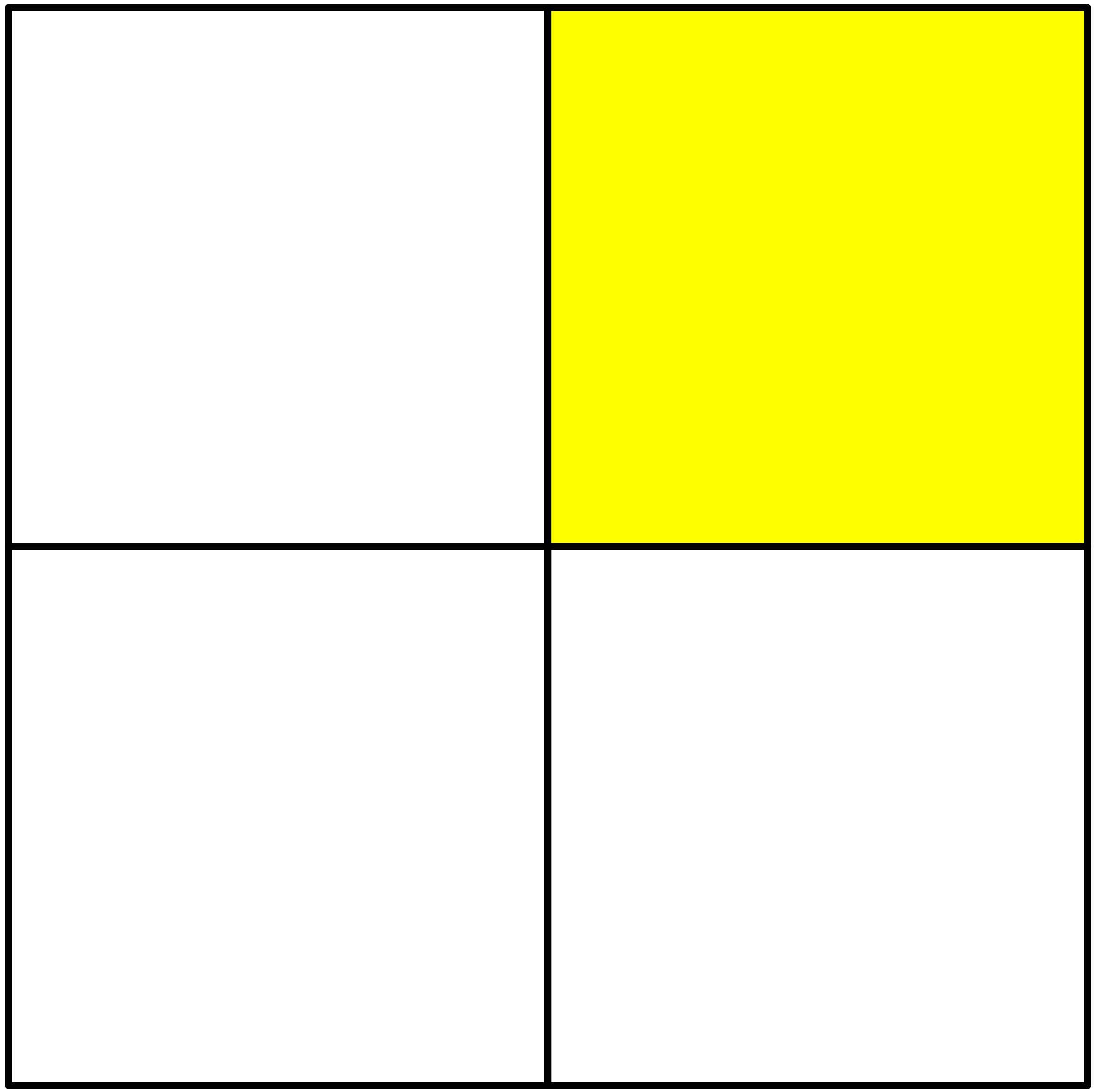
Subdividing a Square



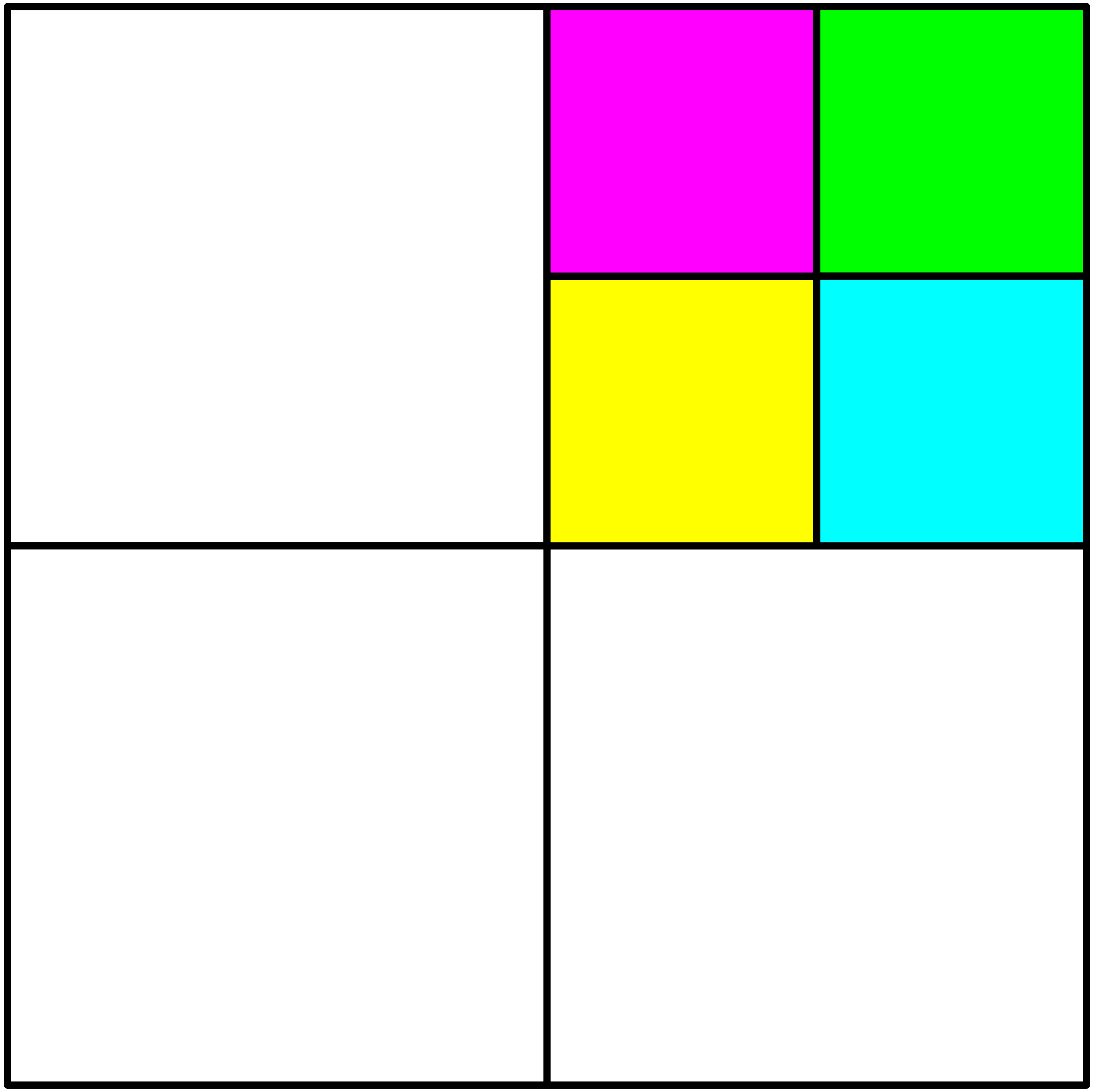
For what values of n can a square be subdivided into n squares?

An Insight

An Insight



An Insight



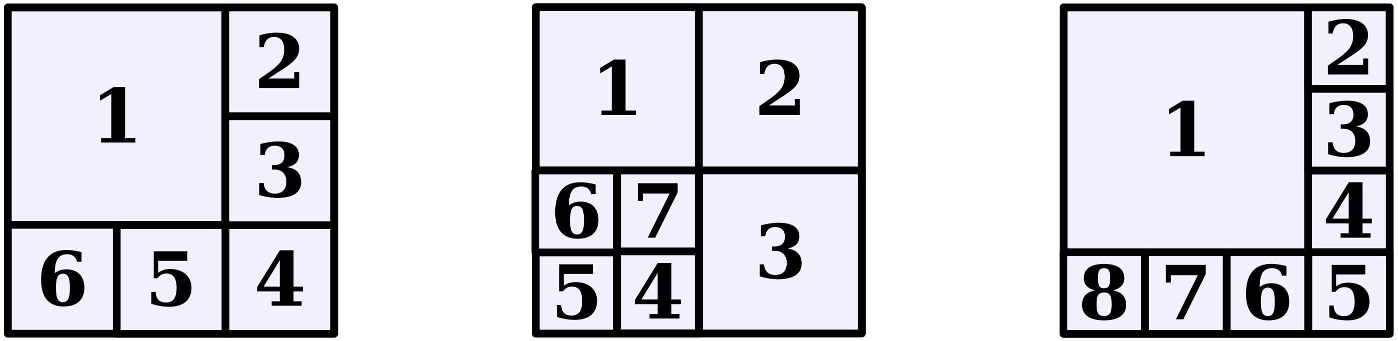
An Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
  - For multiples of three, start with 6 and keep adding three squares until n is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

Theorem: For any $n \geq 6$, there is a way to subdivide a square into n smaller squares.

Proof: Let $P(n)$ be the statement "there is a way to subdivide a square into n smaller squares." We will prove by induction that $P(n)$ holds for all $n \geq 6$, from which the theorem follows.

As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares. This is shown here:



For the inductive step, assume that for some arbitrary $k \geq 6$ that $P(k)$ is true and that there is a way to subdivide a square into k squares. We prove $P(k+3)$, that there is a way to subdivide a square into $k+3$ squares. To see this, start by obtaining (via the inductive hypothesis) a subdivision of a square into k squares. Then, choose any of the squares and split it into four equal squares. This removes one of the k squares and adds four more, so there will be a net total of $k+3$ squares. Thus $P(k+3)$ holds, completing the induction. ■

Generalizing Induction

- When doing a proof by induction,
  - feel free to use multiple base cases, and
  - feel free to take steps of sizes other than one.
- If you do, make sure that…
  - … you actually need all your base cases. Avoid redundant base cases that are already covered by a mix of other base cases and your inductive step.
  - … you cover all the numbers you need to cover. Trace out your reasoning and make sure all the numbers you need to cover really are covered.
- As with a proof by cases, you don't need to separately prove you've covered all the options. We trust you.

More on Square Subdivisions

- There are a ton of interesting questions that come up when trying to subdivide a rectangle or square into smaller squares.
- In fact, one of the major players in early graph theory (William Tutte) got his start playing around with these problems.
- Good starting resource: this Numberphile video on ***Squaring the Square***.

Ramsey Revisited

Ramsey Revisited

- In lecture, we proved the Theorem on Friends and Strangers: any way of painting the edges of K_6 using two colors gives you a monochrome copy of K_3.
- On PS4, you proved that painting the edges of K_{17} using three colors gives you a monochrome copy of K_3.
- What about if you use four colors? Five colors? Six colors? How big does the graph need to be?

Theorem: If $n \geq 1$ is a natural number, then for any way of painting the edges of $K_{3n!}$ with n colors, the graph contains a monochrome K_3.

Proof: Let $P(n)$ be the statement "for all ways of coloring the edges of $K_{3n!}$ with n colors, the graph contains a monochrome K_3." We will prove by induction that $P(n)$ holds for all $n \geq 1$, from which the theorem follows.

As a base case, we prove $P(1)$. Color the edges of K_3 one color; we need to show there is a monochrome K_3. This K_3 *is* a monochrome K_3, so $P(1)$ holds.

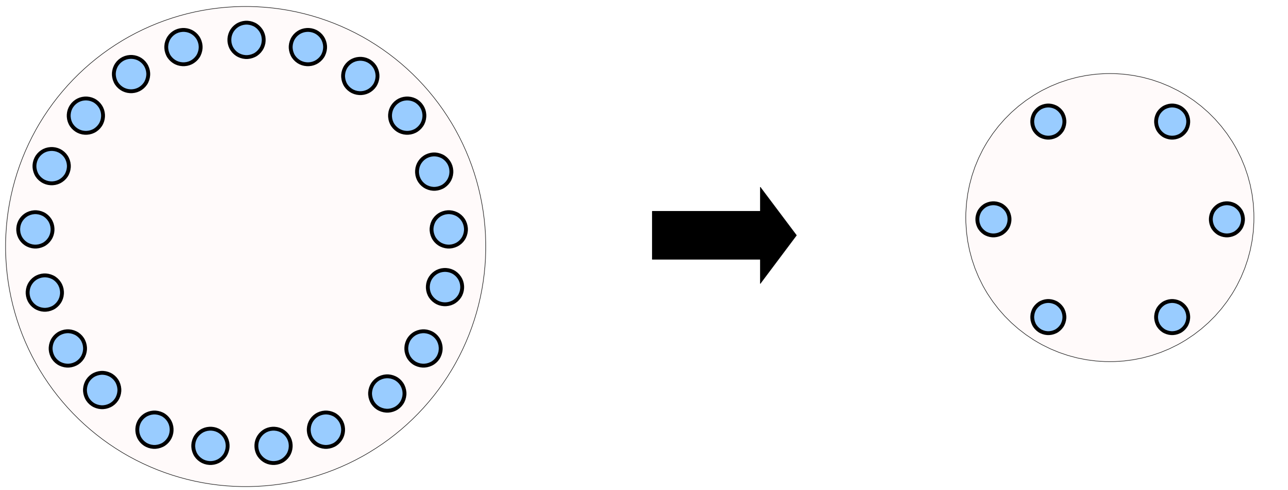
Next, pick a $k \in \mathbb{N}$ where $k \geq 1$ and assume $P(k)$ is true, that any coloring of the edges of $K_{3k!}$ with k colors has a monochrome K_3. We need to show $P(k+1)$ is true. To do so, pick a coloring of the edges $K_{3(k+1)!}$ with $k+1$ colors. We need to find a monochrome K_3.

Pick any node v in $K_{3(k+1)!}$ and look at the edges incident to v. There are $3(k+1)! - 1$ other nodes and $k+1$ colors. By the generalized pigeonhole principle, this means v is adjacent to at least

$$\left\lceil \frac{3(k+1)! - 1}{k+1} \right\rceil = \left\lceil 3k! - \frac{1}{k+1} \right\rceil = 3k!$$

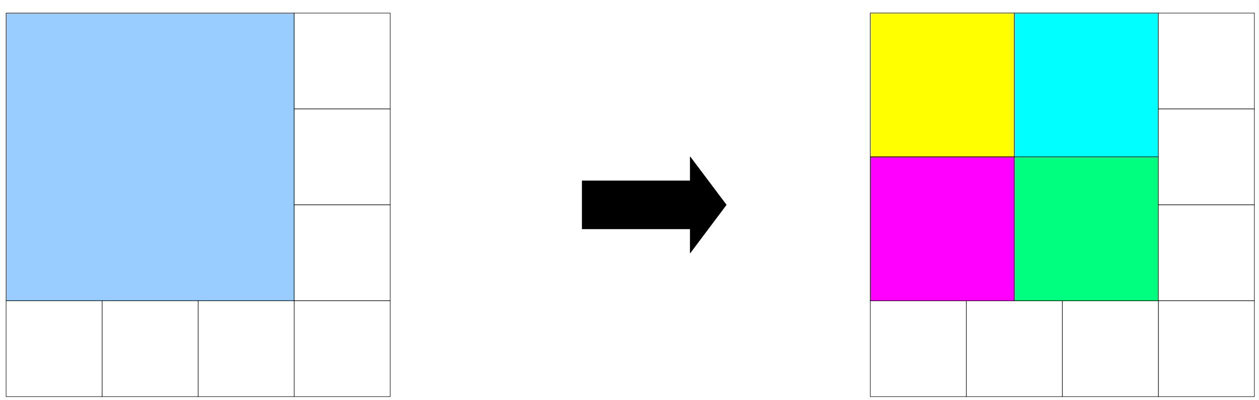
nodes by edges of the same color. Assume WLOG that color is blue. If among those nodes is a blue edge $\{r, s\}$, then v, r, and s form a monochrome K_3. Otherwise, all $3k!$ of those nodes are linked by edges of non-blue colors. We then have a copy of $K_{3k!}$ whose edges are colored using k colors, so by our inductive hypothesis it contains a monochrome K_3. Either way, we find our monochrome K_3, so $P(k+1)$ holds, completing the induction. ■

An Observation



Start with larger graph

Get to smaller graph



Start with fewer squares

Get to more squares

Following the Rules

- When working with square subdivisions, our predicate looked like this:

  $P(n)$ is "***there exists*** a way to subdivide a square into n squares."

- When working with Ramsey theory, our predicate looked like this:

  $P(n)$ is "***for any*** coloring of a $K_{3n!}$, there is a monochrome K_3."

- With squares, the quantifier is ∃. With graphs, the first quantifier is ∀.
- This fundamentally changes the "feel" of induction.

Build Up with $\exists$

- In the case of squares, in our inductive step, we prove

    If

    there exists a subdivision into k squares,

    then

    there exists a subdivision into k+3 squares.

- Assuming the antecedent gives us a concrete subdivision into k squares.
- Proving the consequent means finding some way to subdivide in to k+3 squares.
- The inductive step goal is to "***build up***:" start with a smaller number of squares, and somehow work out what to do to get a larger number of squares.

Build Down with ∀

- In the Ramsey case, in our inductive step, we prove

  If

  for all colorings of $K_{3k!}$, there's a monochrome K_3.

  then

  for all colorings of $K_{3(k+1)!}$, there's a monochrome K_3.

- Assuming the antecedent means once we find a k-colored $K_{3k!}$, we get a monochrome K_3.
- Proving the consequent means picking an arbitrary coloring of $K_{3(k+1)!}$, then trying to find a monochrome K_3 in it.
- The inductive step goal is to "***build down***:" start with a larger graph, then find a way to turn it into a smaller graph.

Some Notes

- Not all predicates P(n) will have the form outlined here.
  - That's okay! Just use the normal rules for assuming and proving things.
  - Think of these as quick shorthands rather than fundamentally new strategies.
- In all cases, assume $P(k)$ and prove $P(k+1)$.
  - All that changes is what you do to assume $P(k)$ and what you do to prove $P(k+1)$.

More on Ramsey Triangles

- We've proved that $3n!$ nodes is enough to get a triangle with $n \geq 1$ colors on the edges.
- For $n = 3$, this says we need 18 nodes, but as you proved on PS4 you can do this with 17 nodes.
- For $n = 4$, this says we need 72 nodes. We know that 50 nodes is too few and 66 nodes is enough. The actual answer is therefore somewhere between 51 and 66.
- ***Open problem:*** Find the exact minimum number of nodes needed to get a monochrome triangle with $n \geq 4$ colors.
- ***Challenge problem:*** Show that $\lceil e \cdot n! \rceil$ nodes is always sufficient to get a monochrome K_3 with $n \geq 1$ colors. *(This is hard but doable if you know the material from CS103, plus the Taylor series for e^x. Come talk to me if you want more details.)*

Time-Out for Announcements!

Problem Set Five

- Problem Set Four was due at 2:30PM today.
- Problem Set Five goes out today. It's due next Friday at 2:30PM.
  - Play around with everything we've covered so far, plus a healthy dose of induction and inductive problem-solving.
- Before starting, read our "Guide to Induction" and "Induction Proofwriting Checklist," which cover common and important cases to look for.
- As always, ping us if you have any questions! That's what we're here for.

Back to CS103!

Complete Induction

Guess what?

It’s time for

Mathematical

Calesthenics!

It’s time for

Mathematicalesthenics!

This is kinda like $P(0)$.

If you are the ***leftmost*** person in your row, stand up right now.

Everyone else: stand up as soon as the person to your left in your row stands up.

This is kinda like $P(k) \to P(k+1)$.

Everyone, please be seated.

Let’s do this again… with a twist!

This is kinda like P(0).

If you are the ***leftmost*** person in your row, stand up right now.

Everyone else: stand up as soon as ***everyone*** left of you in your row stands up.

What sort of sorcery is this?

Let P be some predicate. The ***principle of complete induction*** states that if

$P(0)$ is true

and

for all $k \in \mathbb{N}$, if $P(0)$, …, and $P(k)$ are true, then $P(k+1)$ is true

then

$\forall n \in \mathbb{N}.\ P(n)$

If it starts true…

…and it stays true…

…then it's always true.

Mathematical Induction

- You can write proofs using the principle of mathematical induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $P(k)$ is true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

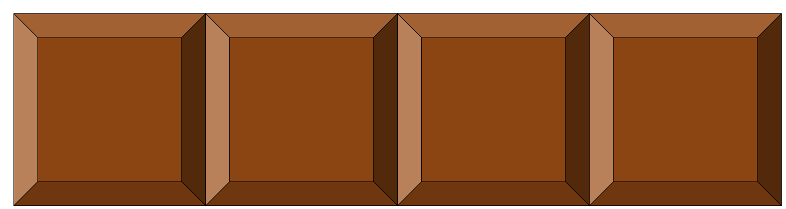
Complete Induction

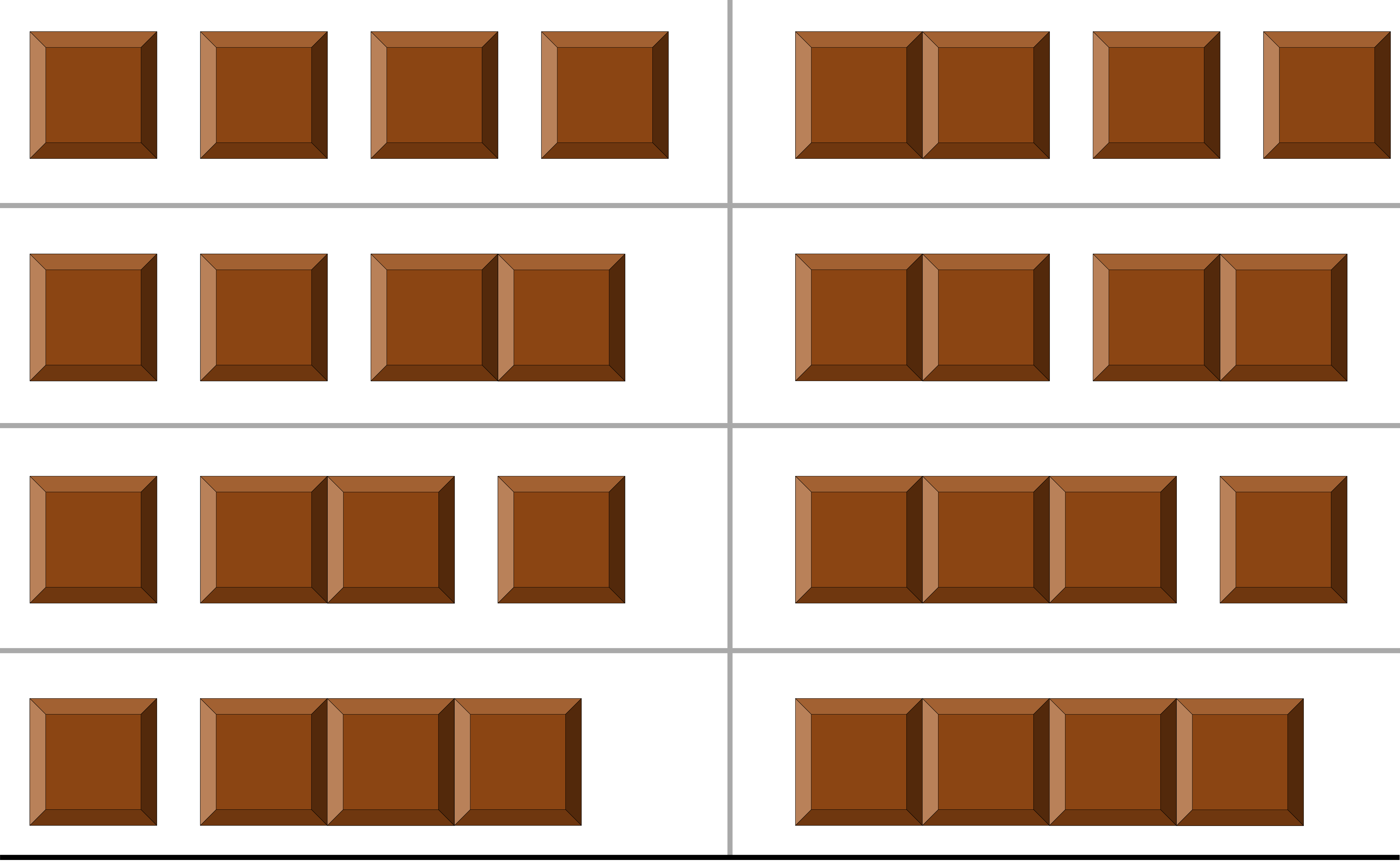
- You can write proofs using the principle of ***complete*** induction as follows:
  - Define some predicate $P(n)$ to prove by induction on n.
  - Choose and prove a base case (probably, but not always, $P(0)$).
  - Pick an arbitrary $k \in \mathbb{N}$ and assume that $\boldsymbol{P(0), P(1), P(2), \ldots,}$ **and** $\boldsymbol{P(k)}$ are all true.
  - Prove $P(k+1)$.
  - Conclude that $P(n)$ holds for all $n \in \mathbb{N}$.

An Example: ***Eating a Chocolate Bar***

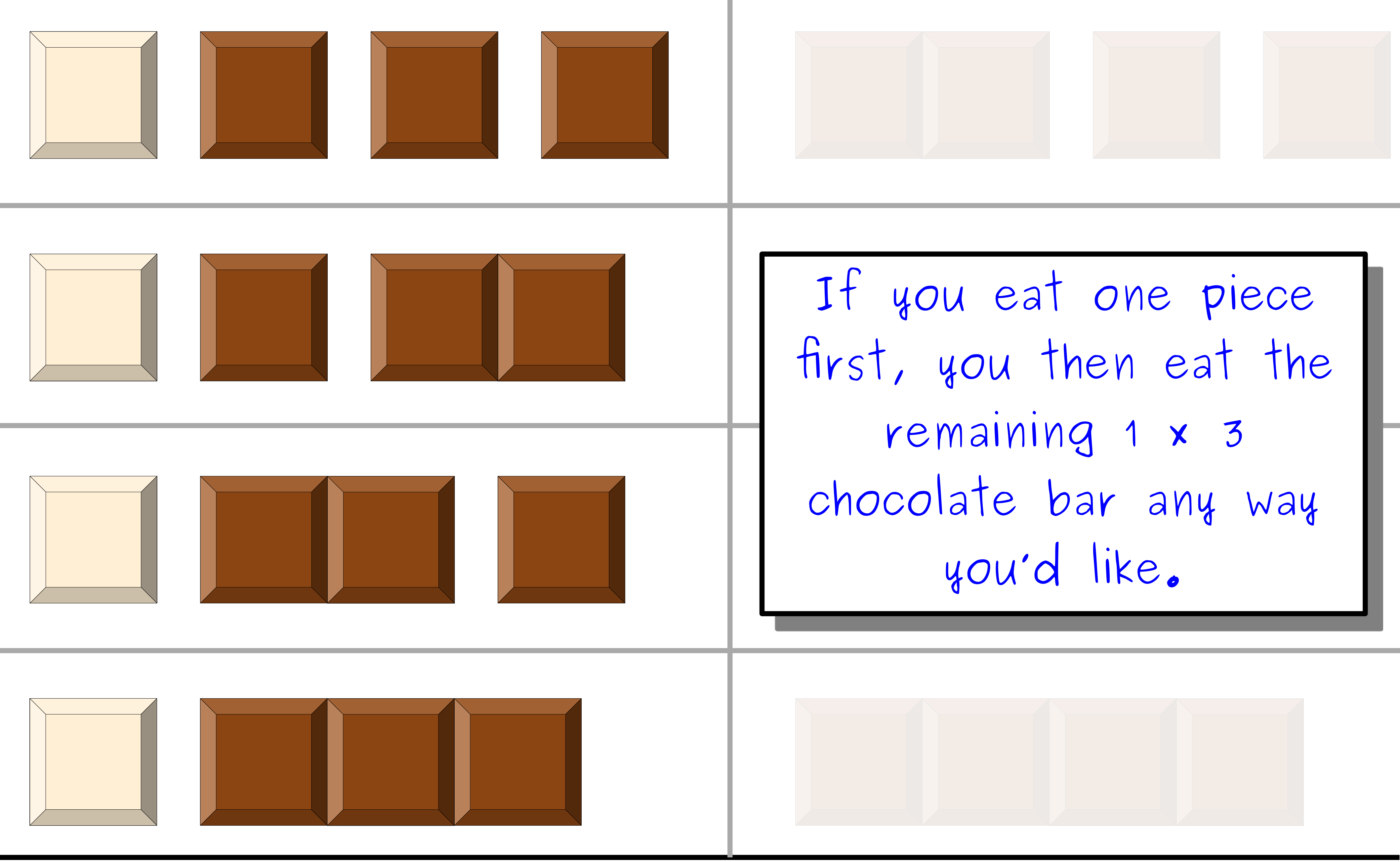
Eating a Chocolate Bar

- You have a $1 \times n$ chocolate bar subdivided into 1×1 squares.
- You eat the chocolate bar from left to right by breaking off one or more squares and eating them in one (possibly enormous) bite.
- How many ways can you eat a…
  - 1×1 chocolate bar?
  - 1×2 chocolate bar?
  - 1×3 chocolate bar?
  - 1×4 chocolate bar?





There are eight ways to eat a 1×4 chocolate bar.



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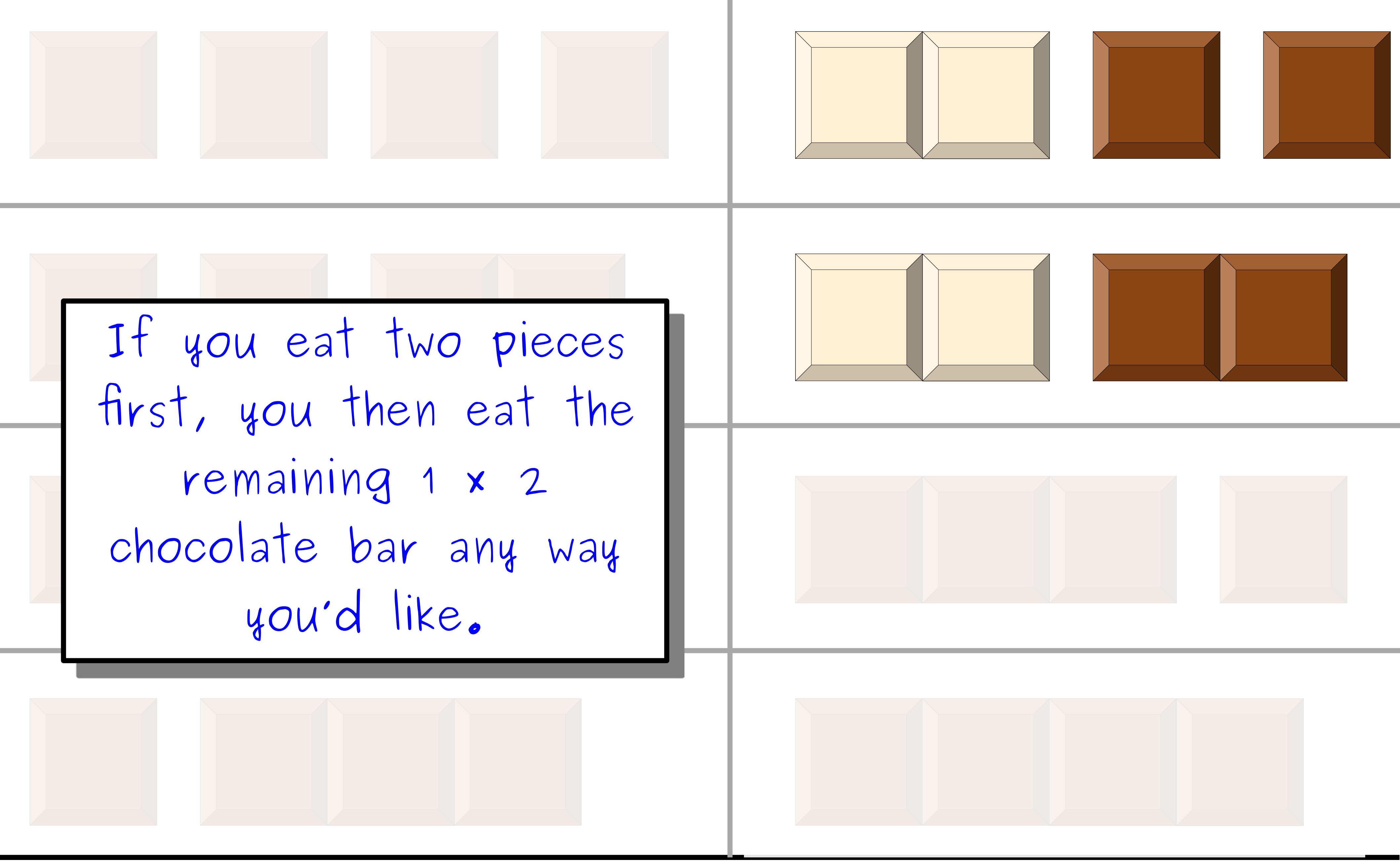
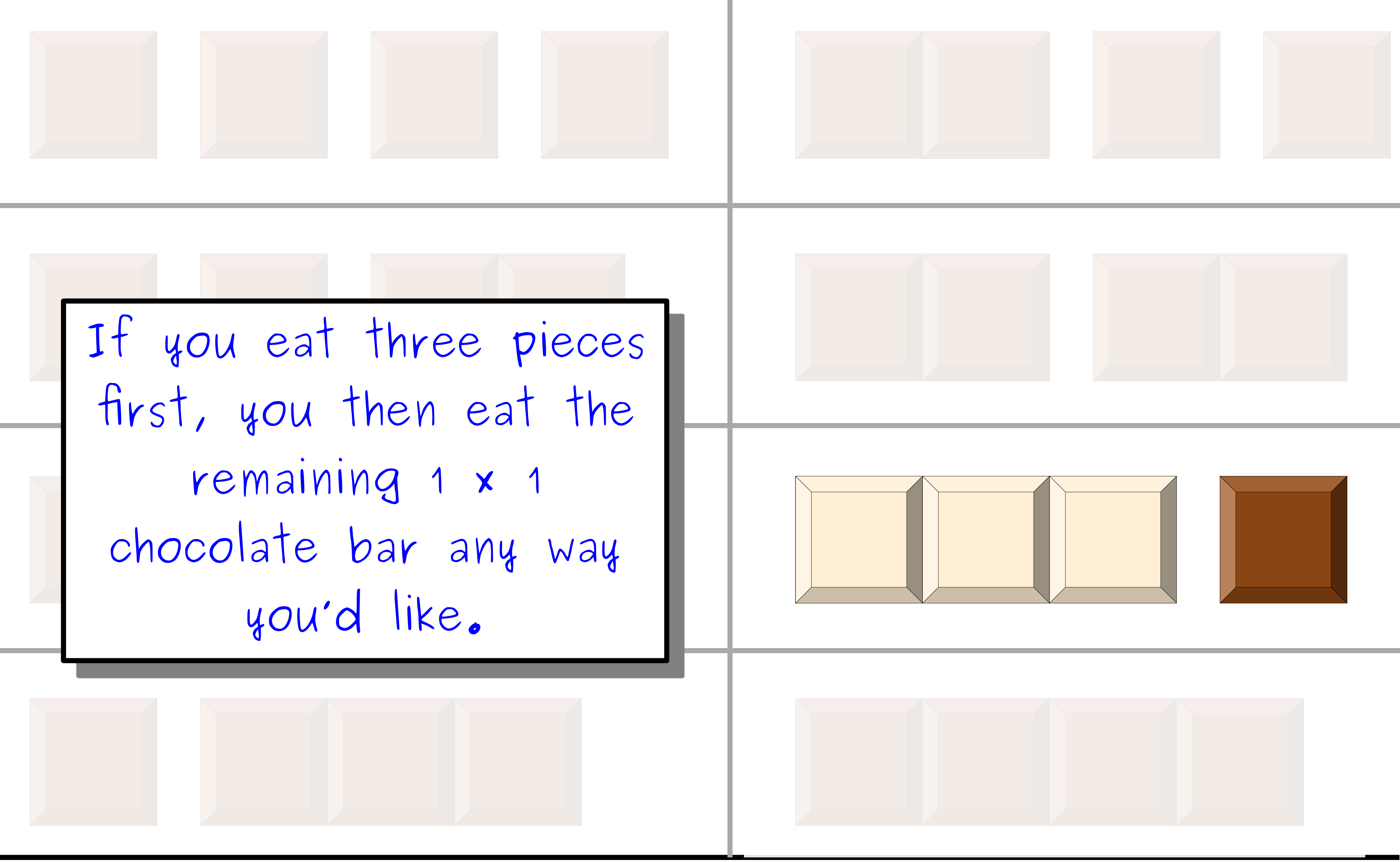


There are eight ways to eat a 1 × 4 chocolate bar.



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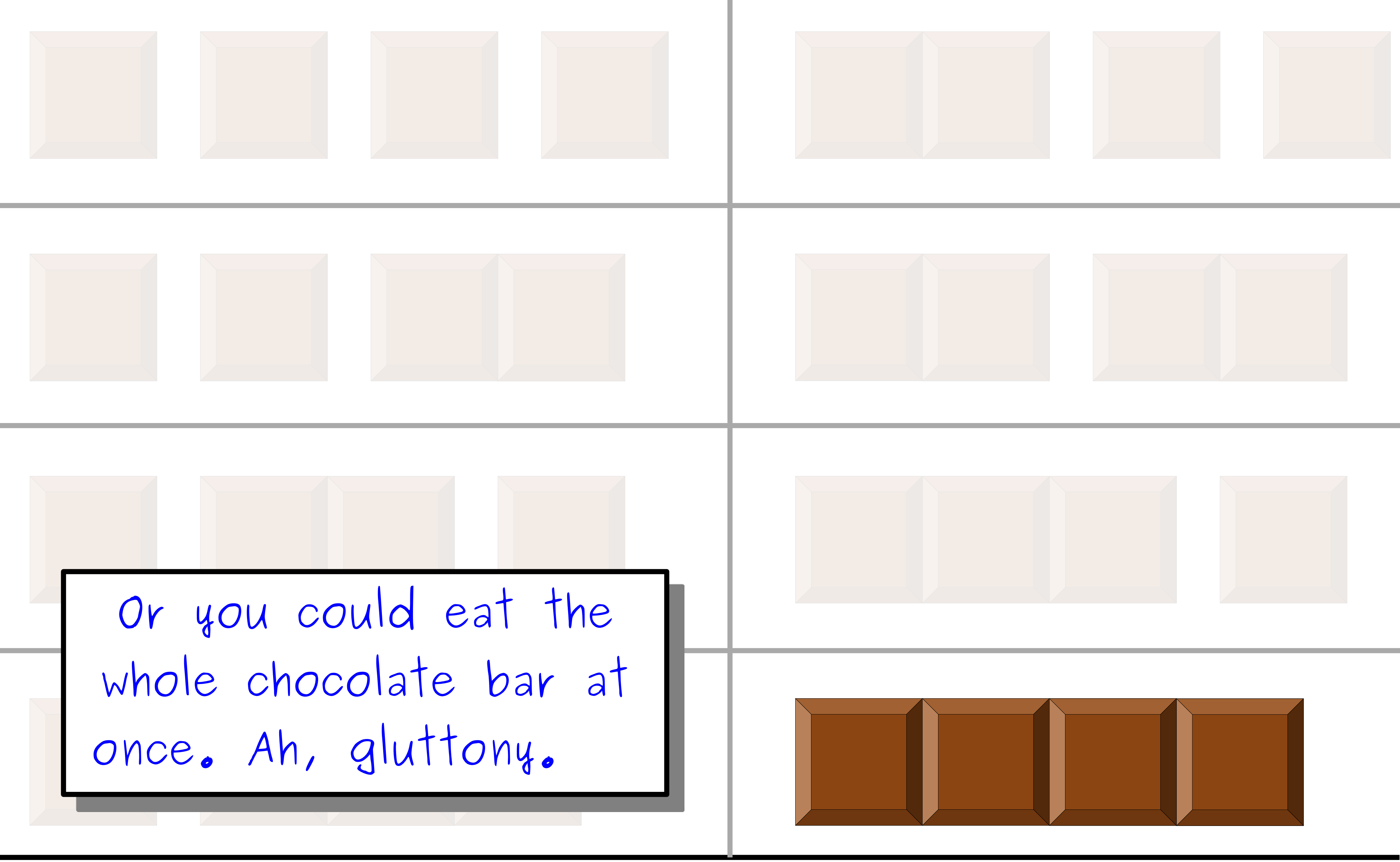


There are eight ways to eat a 1 × 4 chocolate bar.

Eating a Chocolate Bar

- There's…
  - 1 way to eat a 1×1 chocolate bar,
  - 2 ways to eat a 1×2 chocolate bar,
  - 4 ways to eat a 1×3 chocolate bar, and
  - 8 ways to eat a 1×4 chocolate bar.
- ***Our guess:*** There are 2^{n-1} ways to eat a $1 \times n$ chocolate bar for any natural number $n \geq 1$.
- And we think it has something to do with this insight: we eat the bar either by
  - eating the whole thing in one bite, or
  - eating some piece of size k, then eating the remaining $n - k$ pieces however we'd like.
- Let's formalize this!

Theorem: For any natural number $n \geq 1$, there are exactly 2^{n-1} ways to eat a $1 \times n$ chocolate bar from left to right.

Proof: Let $P(n)$ be the statement "there are exactly 2^{n-1} ways to eat a $1 \times n$ chocolate bar from left to right." We will prove by induction that $P(n)$ holds for all natural numbers $n \geq 1$, from which the theorem follows.

As our base case, we prove $P(1)$, that there is exactly $2^{1-1} = 1$ way to eat a 1×1 chocolate bar from left to right. The only option here is to eat the entire chocolate bar at once, so there's just one way to eat it, as needed.

For our inductive step, assume for some arbitrary natural number $k \geq 1$ that $P(1)$, …, and $P(k)$ are true. We need to show $P(k+1)$ is true, that there are exactly 2^k ways to eat a $1 \times (k+1)$ chocolate bar.

There are two options for how to eat the bar. First, we can eat the whole chocolate bar in one bite. Second, we could eat a piece of size r for some $1 \leq r \leq k$, leaving a chocolate bar of size $k+1-r$, then eat that chocolate bar from left to right. Since $1 \leq r \leq k$, we know that $1 \leq k+1-r \leq k$, so by our inductive hypothesis there are 2^{k-r} ways to eat the remainder.

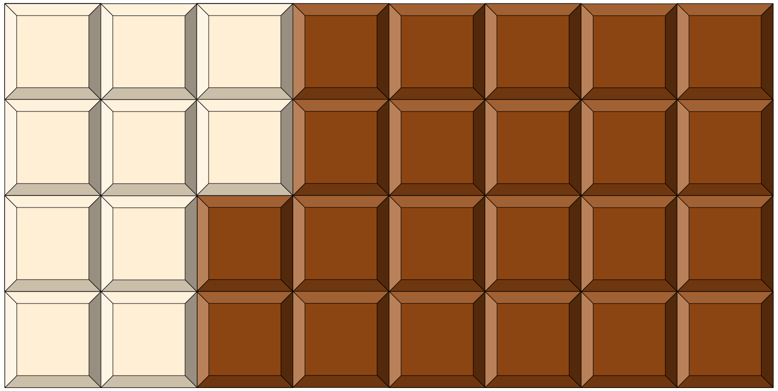
Summing up this first option, plus all choices of r for the second option, we see that the number of ways to eat the chocolate bar is

$$1 + 2^{k-1} + 2^{k-2} + \ldots + 2^2 + 2^1 + 2^0 \quad = \quad 1 + 2^k - 1 \quad = \quad 2^k.$$

Thus $P(k+1)$ holds, completing the induction. ■

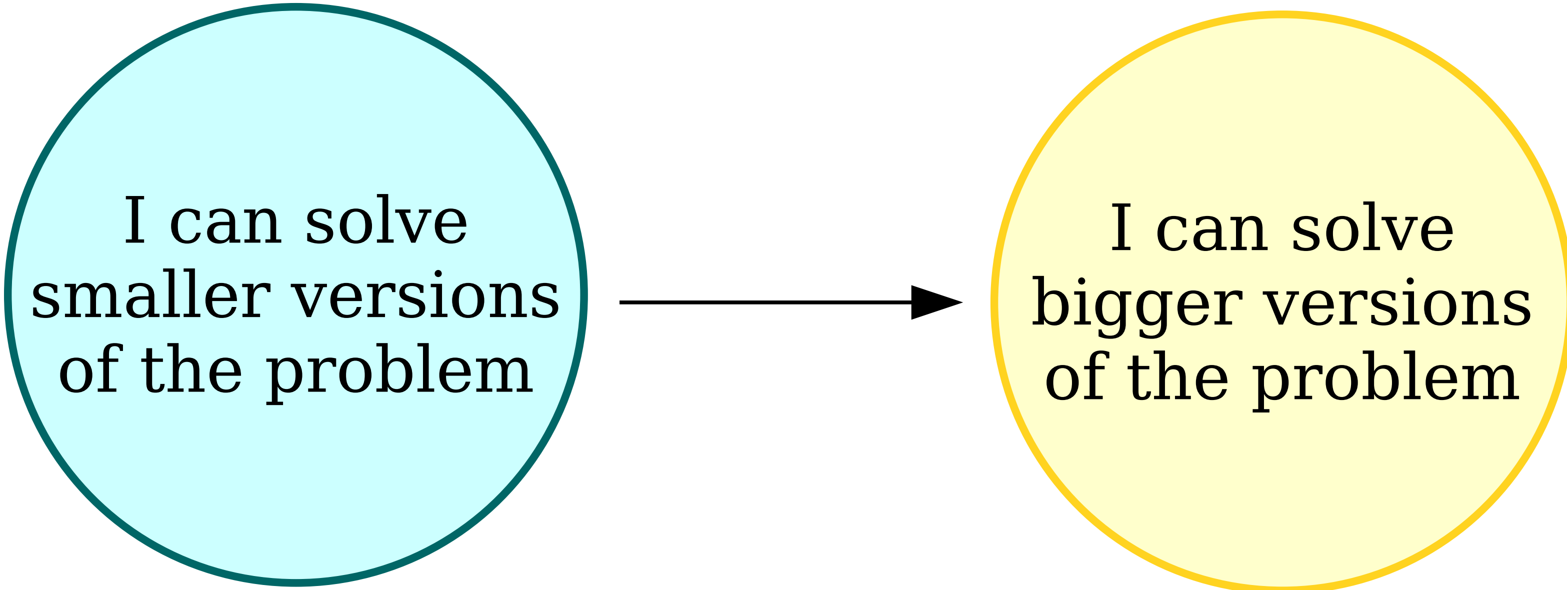
More on Chocolate Bars

- Imagine you have an $m \times n$ chocolate bar. Whenever you eat a square, you have to eat all squares above it and to the left.
- How many ways are there to eat the chocolate bar?



- ***Open Problem:*** Find a non-recursive exact formula for this number, or give an approximation whose error drops to zero as m and n tend toward infinity.

Induction vs. Complete Induction



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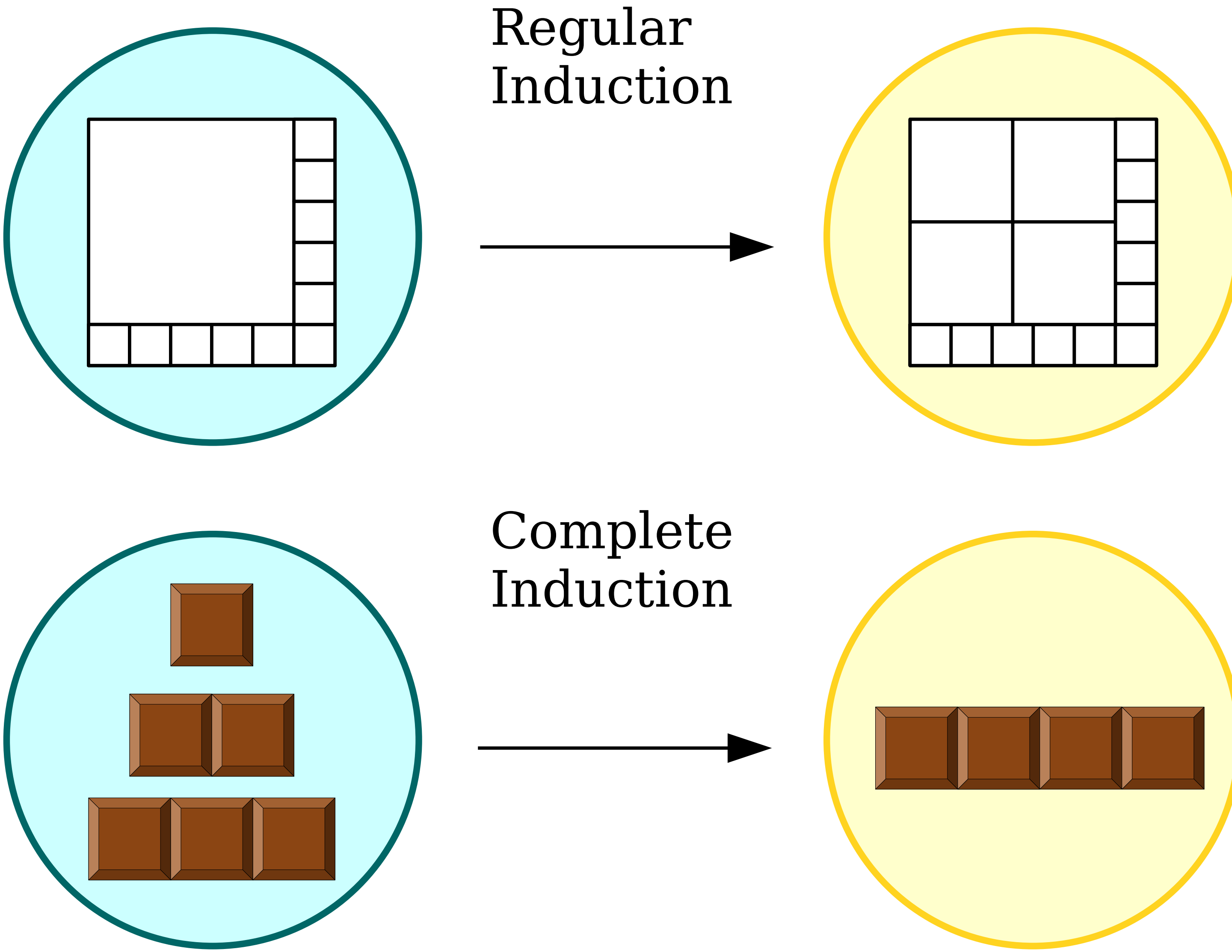
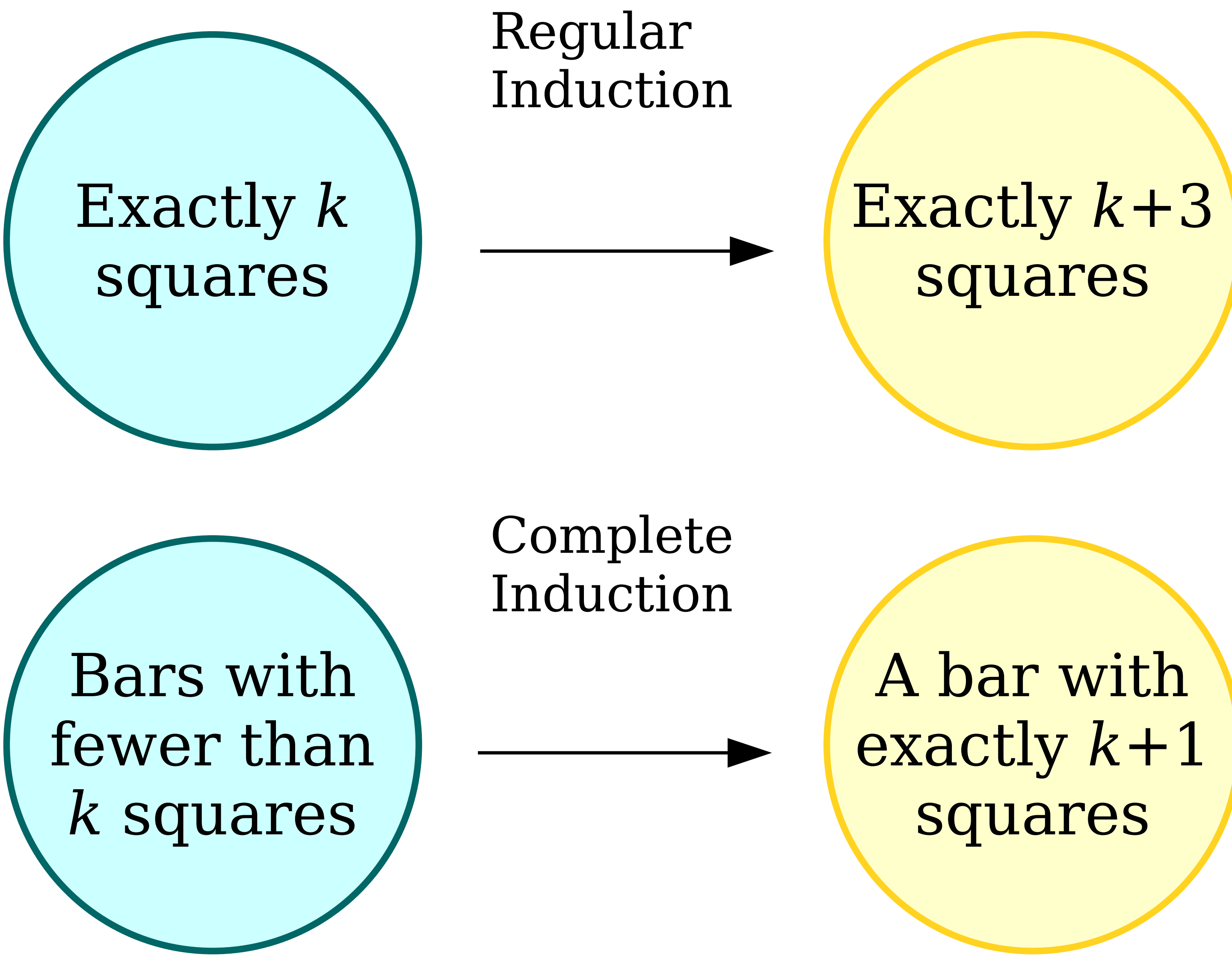
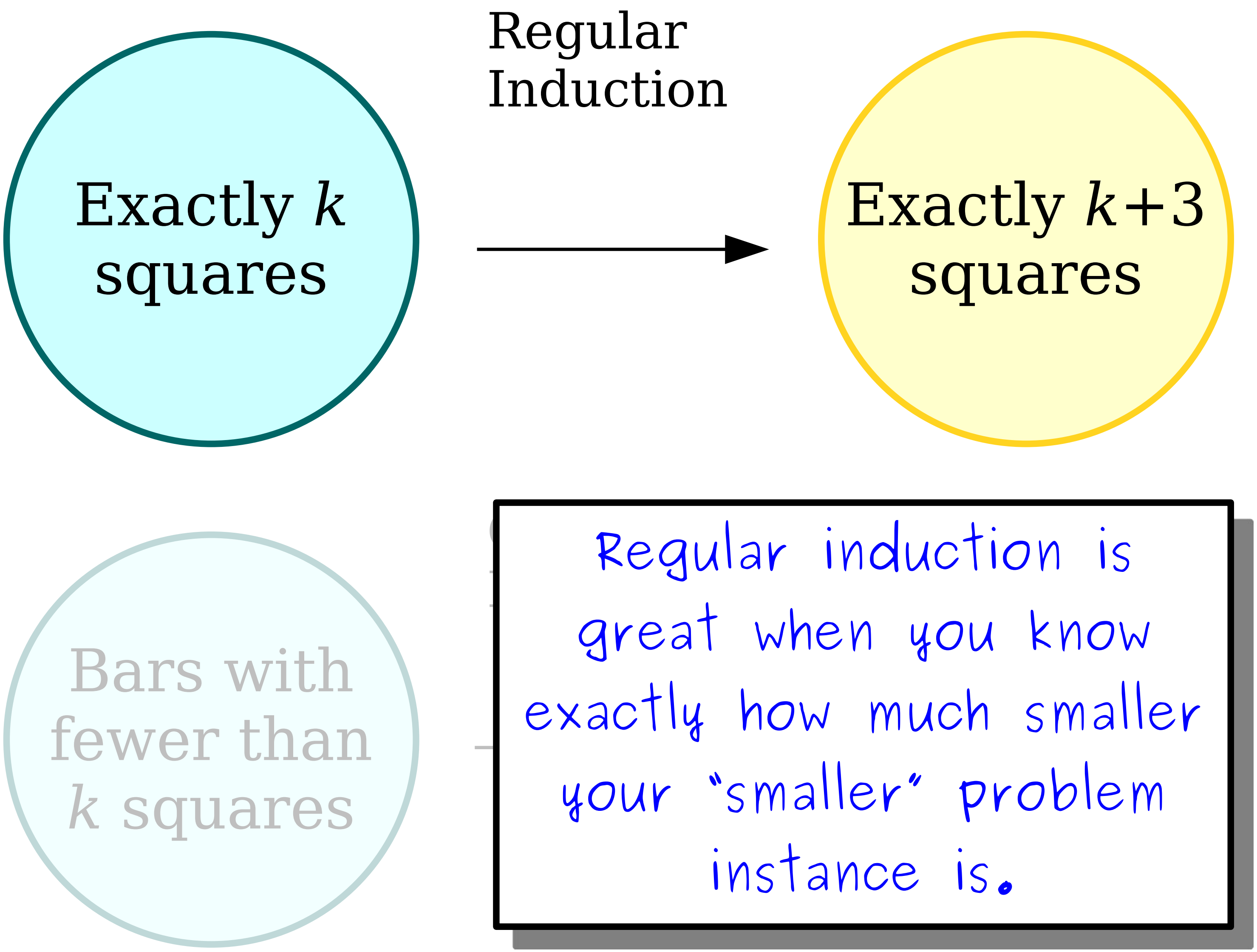


Induction vs. Complete Induction

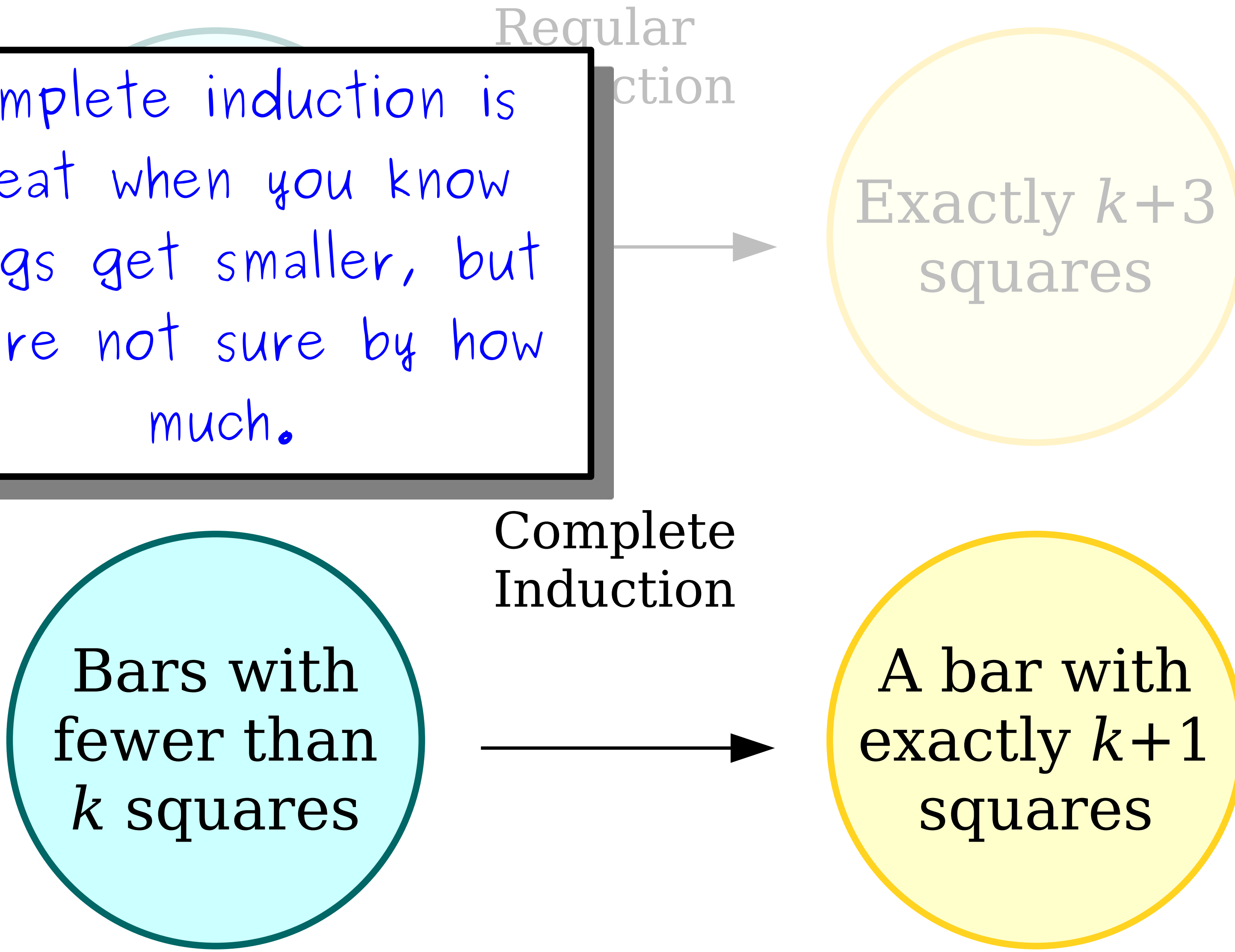


Induction vs. Complete Induction



Induction vs. Complete Induction

Complete induction is great when you know things get smaller, but you're not sure by how much.



An Important Milestone

Recap: ***Discrete Mathematics***

- The past five weeks have focused exclusively on discrete mathematics:

Induction	Functions
Graphs	The Pigeonhole Principle
Formal Proofs	Mathematical Logic
Set Theory	Cardinality

- These are building blocks we will use throughout the rest of the quarter.
- These are building blocks you will use throughout the rest of your CS career.

Next Up: ***Computability Theory***

- It's time to switch gears and address the limits of what can be computed.
- We'll explore these questions:
  - How do we model computation itself?
  - What exactly is a computing device?
  - What problems can be solved by computers?
  - What problems *can't* be solved by computers?
- ***Get ready to explore the boundaries of what computers could ever be made to do.***

Next Time

- ***Formal Language Theory***
  - How are we going to formally model computation?
- ***Finite Automata***
  - A simple but powerful computing device made entirely of math!
- ***DFAs***
  - A fundamental building block in computing.